Note Title 1/19/2012

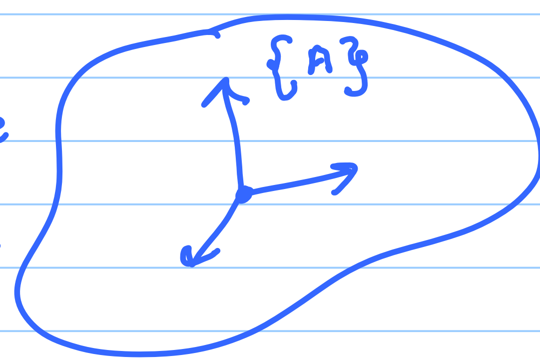
Basic KINEMATICS

1) Represent rigid body motion → trans.
Rotation

2) articulated arms

3) velocities

attach a frame
to the body.



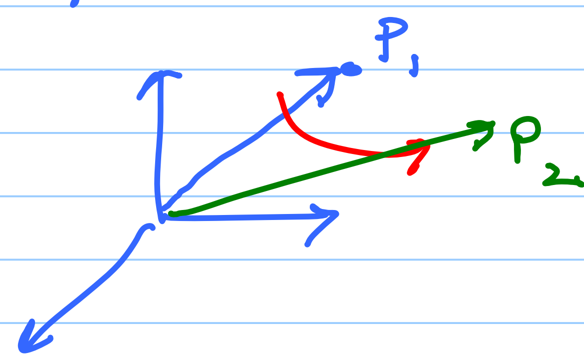
Motions of the frame

① Translation $\xrightarrow[N\times 1]{}$ vectors in $\mathbb{R}^N$

② Rotations $\xrightarrow[N\times N]{}$ $N\times N$ matrices

3 interpretations: (pure rotations)

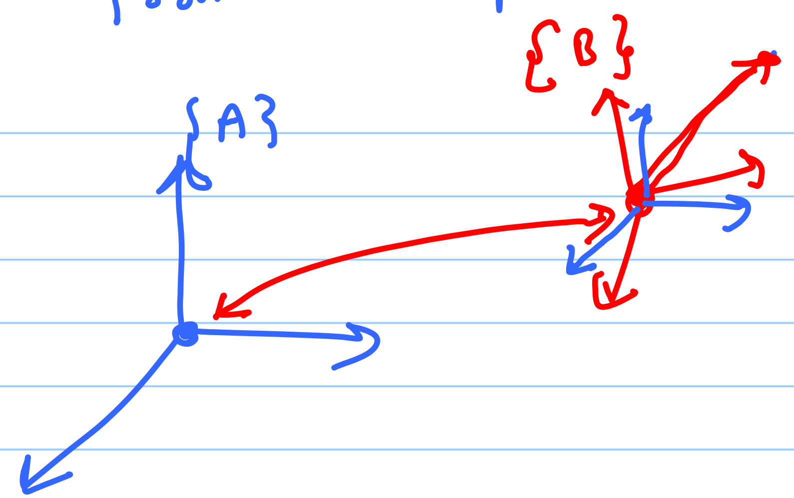
① operator: rotate points/vectors w.r.t. a fixed frame.



$P_2 = R\,P_1$

② as a mapping of the $\boxed{\text{same pt.}}$

from one frame to another



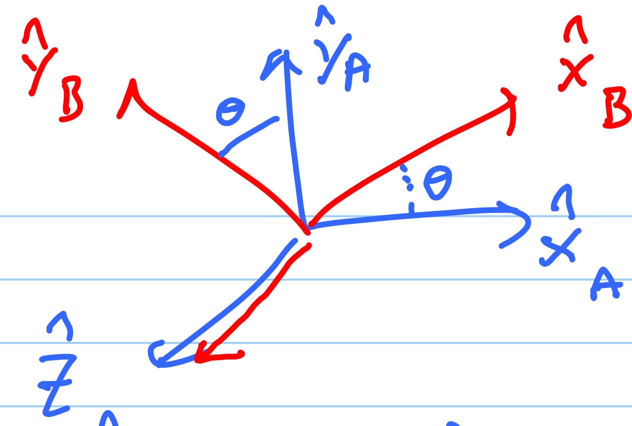
$${}^{B}P = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$${}^{A}P = {}^{A}_{B}R \, {}^{B}P$$

(3) represents a cord. frame w. r. t. another

$${}^{A}_{B}R = \text{rep. of } \{B\} \text{ wrt } \{A\}$$

$${}^{A}_{B}R = \left[{}^{A}\hat{X}_B \;\vdots\; {}^{A}\hat{Y}_B \;\vdots\; {}^{A}\hat{Z}_B \right]$$



$$
{}^{A}_{B}R = \begin{array}{ccc} {}^{A}\hat{x}_B & {}^{A}\hat{y}_B & {}^{A}\hat{z}_B \\ \left[\begin{array}{c|c|c} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{array}\right. & & \left.\right] \end{array}
$$

$R^T = R^{-1}$

<u>prop. of Rot. matrices:</u> ① Col. vect. are unit vect

N-dim ② " " " mutually $\perp$

"orthonormal matrices"

3 $\det(R) = +1$

3x3 matrices $\longrightarrow$ 9 elements

Space of rot. matrices as $\mathbb{R}^9$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ \vdots & & \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

but constraints : $\left\{ \begin{array}{l} r_{11}^2 + r_{21}^2 + r_{31}^2 = 1 \\ r_{11} \cdot r_{21} + r_{12} \cdot r_{22} + r_{13} \cdot r_{23} = 0 \end{array} \right.$

6 constraints

$\Downarrow$

3 independent elements !!

$\Longrightarrow$ 3 dim manifold.

$SO(3)$ or $SO(N)$

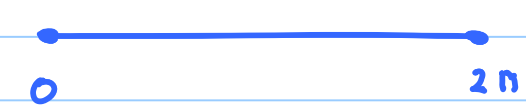
rotations in plane:

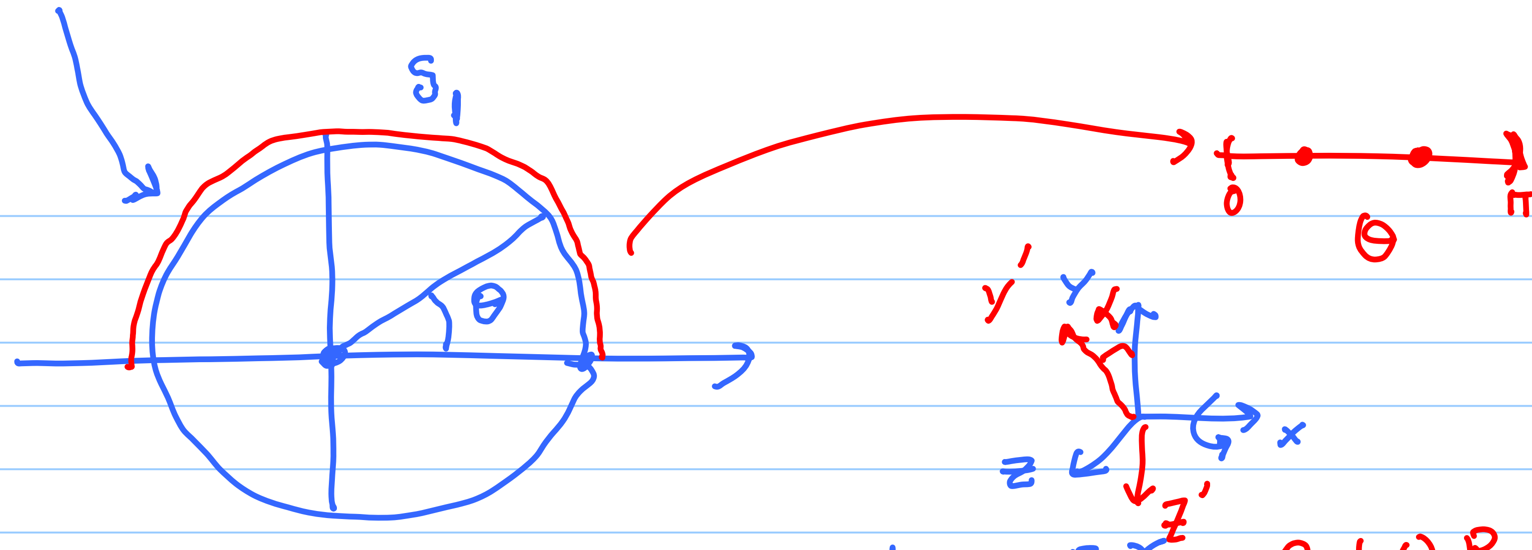
$$\begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}$$

$SO(2)$:

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

$$\begin{cases} r_{11}^2 + r_{12}^2 = 1 \\ r_{21}^2 + r_{22}^2 = 1 \\ r_{11} \cdot r_{12} + r_{21} \cdot r_{22} = 0 \end{cases}$$





$SO(3)$

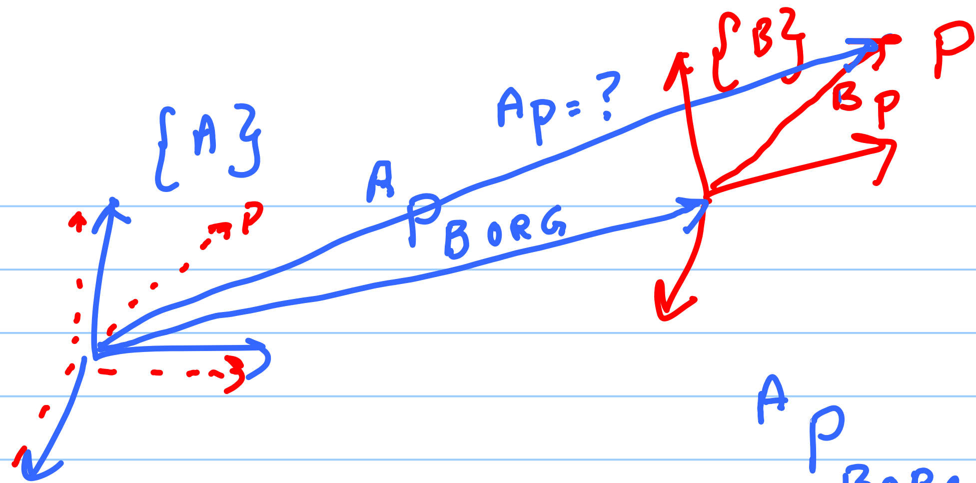
Euler angles α, β, γ

$R_z(\alpha) R_y(\beta) R_z(\gamma)$

$R_{\hat{k}}(\theta)$

$\underline{k_x, k_y, \theta}$

fixed angle



$$\underbrace{{}^{A}P_{BORG}\,,\ {}^{A}_{B}R}$$

$$ {}^{A}P \stackrel{?}{=} {}^{B}P $$

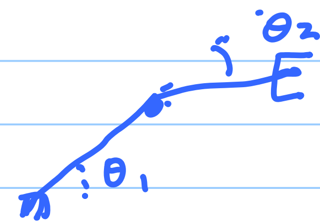
$$ {}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] $$

$$ = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

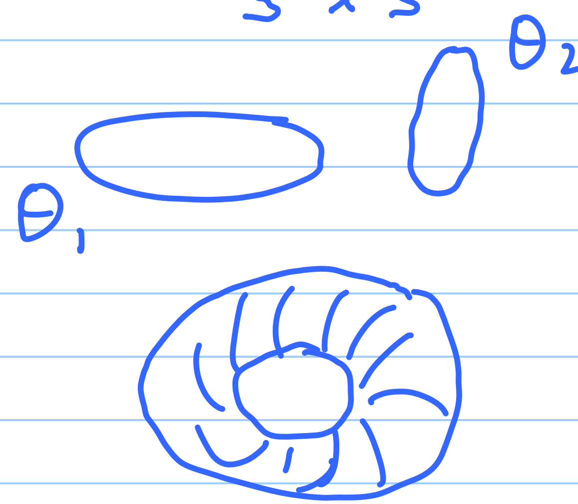
SE(3)

$$ \begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix} $$

$$\boxed{\mathbb{R}^3 \times SO(3)}$$



$S^1 \times S^1$



Velocities: inst. motion

linear vel, angular vel.

$\downarrow$ point (vector) ✓

$\downarrow$ frame $\downarrow$ (vectors)

$SO(N)$ or $SO(3)$

$$R R^T = I$$

$$\dot{R} R^T + R \dot{R}^T = 0$$

$$\Rightarrow \dot{R} R^T = -R \dot{R}^T$$

$$\shortparallel$$

$$S = -(\dot{R} R^T)^T = -S^T$$

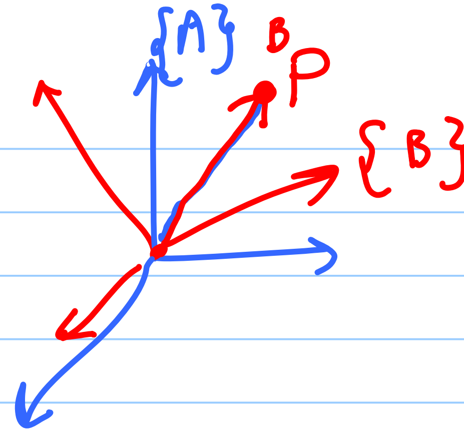
$$v_p = \dot{R} P$$

$$^A P = {}^A_B R \, {}^B P$$

$$^A \dot{P} = {}^A_B \dot{R} \, {}^B P$$

$$\Omega$$

$$\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}$$



$$= \boxed{{}^A_B\dot{R}\left({}^A_B R\right)^T {}^A P} + \cancel{{}^A_B R\,{}^B\dot{P}} \quad (*)$$

$${}^A P = {}^A_B R\,{}^B P$$

$$\left[{}^A_B R\right]^T {}^A P = {}^B P$$

$${}^B_A R\,{}^A P = {}^B P$$

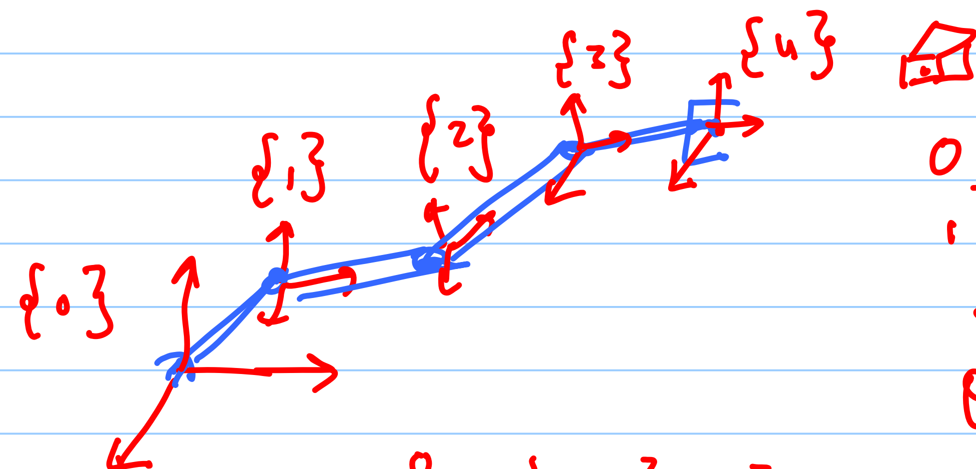
$${}^B P = {}^B_A R\,{}^A P$$

$${}^A\Omega_B \qquad {}^A_B S \cdot {}^A P$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times {}^A P$$

$$\circledast \quad {}^{A}\dot{P} = {}^{A}\Omega_{B} \times {}^{A}P + {}^{A}_{B}R\,{}^{B}\dot{P}$$

$${}^{A}v_{P} = {}^{A}\Omega_{B} \times {}^{A}P + {}^{A}_{B}R\,{}^{B}v_{P}$$



$${}^{0}_{1}T,\quad {}^{1}_{2}T,\quad {}^{2}_{3}T,\quad {}^{3}_{4}T$$

$$\theta_1 \qquad \theta_2 \qquad \theta_3$$

$${}^{0}_{4}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T = \begin{bmatrix} {}^{0}_{4}R & {}^{0}P_{4org} \\ 0\;0\;0 & 1 \end{bmatrix}$$

$\theta_1, \theta_2, \theta_3, \theta_4$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} \underset{\text{inv.}}{\overset{\text{forw}}{\rightleftarrows}} \underline{\mathbb{R}}^3 \times \underline{SO(3)}$$